Homework 2 Sample Solutions

provided by Kenneth Co

Extra Exercise 1. (a) Show that the following decomposition cannot hold

$$\frac{x^3 + 10x^2 + 3x + 36}{(x-1)(x^2+4)^2} = \frac{A}{x-1} + \frac{Bx + C}{(x^2+4)^2}$$

for constants A, B, and C.

(b) Show that the following decomposition cannot hold

$$\frac{5x-7}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{x-1} + \frac{C}{x-1}$$

for constants A, B, and C.

Solution.

(a) Notice that when we add the right-hand side using their least common denominator, their sum will not have the term x^3 in the numerator.

$$\begin{aligned}\frac{A}{x-1} + \frac{Bx+C}{(x^2+4)^2} &= \frac{A(x^2+4)^2}{(x-1)(x^2+4)^2} + \frac{(Bx+C)(x-1)}{(x-1)(x^2+4)^2} \\ &= \frac{Ax^4 + 8Ax^2 + 16A}{(x-1)(x^2+4)^2} + \frac{Bx^2 + Cx - Bx - C}{(x-1)(x^2+4)^2}\end{aligned}$$

So for the following decomposition

$$\frac{x^3 + 10x^2 + 3x + 36}{(x-1)(x^2+4)^2} = \frac{A}{x-1} + \frac{Bx + C}{(x^2+4)^2}$$

to hold, we must have the following equality

$$x^3 + 10x^2 + 3x + 36 = Ax^4 + 8Ax^2 + 16A + Bx^2 + Cx - Bx - C.$$

However, the left-hand side has x^3 with a coefficient of one, whereas the right-hand side has x^3 with a coefficient of zero regardless of the constants A, B, and C. Thus, the decomposition cannot hold.

(b) Suppose the given decomposition is true for some constants A, B, and C. We will now look for a contradiction.

$$\begin{aligned}
\frac{5x-7}{(x-1)^3} &= \frac{A}{x-1} + \frac{B}{x-1} + \frac{C}{x-1} \\
&= \frac{A(x-1)^2}{(x-1)^3} + \frac{B(x-1)^2}{(x-1)^3} + \frac{C(x-1)^2}{(x-1)^3} \\
&= (A+B+C)\frac{(x-1)^2}{(x-1)^3} \\
&= \frac{(A+B+C)x^2 - 2(A+B+C)x + (A+B+C)}{(x-1)^3}
\end{aligned}$$

This tells us that $A+B+C=0$, $2(A+B+C)=5$, and $A+B+C=-7$. These clearly can't be all true, so we have our contradiction. Thus, the decomposition cannot hold.

□

Exercise 7.2 #36. (a) Use integration by parts to show that

$$\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx.$$

(b) Apply the reduction formula in (a) repeatedly to compute

$$\int x^3 e^x \, dx.$$

Solution.

(a) We set $u = x^n$ and $dv = e^x$. Then, $du = nx^{n-1}\, dx$ and $v = e^x$. The desired integral follows from integration by parts.

(b) Using (a) several times, we get the following equations.

$$\begin{aligned}
\int x^3 e^x \, dx &= x^3 e^x - 3 \int x^2 e^x \, dx \\
&= x^3 e^x - 3(x^2 e^x - 2 \int x e^x \, dx) \\
&= x^3 e^x - 3x^2 e^x + 6(x e^x - \int e^x \, dx) \\
&= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C \\
&= e^x(x^3 - 3x^2 + 6x - 6) + C
\end{aligned}$$

□

Exercise 7.2 #64. Simplify the integrand and then use an appropriate substitution to evaluate

$$\int \frac{\sin^2 x - \cos^2 x}{(\sin x - \cos x)^2}\, dx.$$

Solution.

$$\begin{aligned}\int \frac{\sin^2 x - \cos^2 x}{(\sin x - \cos x)^2}\, dx &= \int \frac{(\sin x - \cos x)(\sin x + \cos x)}{(\sin x - \cos x)^2}\, dx \\ &= \int \frac{\sin x + \cos x}{\sin x - \cos x}\, dx\end{aligned}$$

We make the substitution $u = \sin x - \cos x$, to get $du = (\sin x + \cos x)\, dx$ and

$$\int \frac{\sin x + \cos x}{\sin x - \cos x}\, dx = \int \frac{1}{u}\, du = \ln|u| + C = \ln|\sin x - \cos x| + C.$$

□

Exercise 7.3 #16. Use partial-fraction decomposition to evaluate the integral

$$\int \frac{1}{(x-1)(x+2)}\, dx.$$

Solution.

$$\begin{aligned}\frac{1}{(x-1)(x+2)} &= \frac{A}{x-1} + \frac{B}{x+2} \\ &= \frac{A(x+2)}{(x-1)(x+2)} + \frac{B(x-1)}{(x-1)(x+2)} \\ &= \frac{Ax + 2A + Bx - B}{(x-1)(x+2)} \\ &= \frac{(A+B)x + (2A - B)}{(x-1)(x+2)}\end{aligned}$$

This tells us that $A + B = 0$ and $2A - B = 1$. Solving for their values, we get $A = \frac{1}{3}$ and $B = -\frac{1}{3}$. Finally, we compute for the integral.

$$\begin{aligned}\int \frac{1}{(x-1)(x+2)}\, dx &= \int \frac{1}{3(x-1)}\, dx - \int \frac{1}{3(x+2)}\, dx \\ &= \frac{1}{3}(\ln|x-1| - \ln|x+2|) + C\end{aligned}$$

□

Exercise 7.3 #18. Use partial-fraction decomposition to evaluate the integral

$$\int \frac{4x^2 - x - 1}{(x+1)^2(x-3)}\, dx.$$

Solution.

$$\begin{aligned}
\frac{4x^2-x-1}{(x+1)^2(x-3)} &= \frac{A}{x-3}+\frac{B}{x+1}+\frac{C}{(x+1)^2} \\
&= \frac{A(x+1)^2}{(x+1)^2(x-3)}+\frac{B(x-3)(x+1)}{(x+1)^2(x-3)}+\frac{C(x-3)}{((x+1)^2(x-3)} \\
&= \frac{Ax^2+2Ax+A+Bx^2-2Bx-3B+Cx-3C}{(x+1)^2(x-3)} \\
&= \frac{(A+B)x^2+(2A-2B+C)x+(A-3B-3C)}{(x+1)^2(x-3)}
\end{aligned}$$

This tells us that

$$A+B=4, \quad 2A-2B+C=-1, \quad \text{and} \quad A-3B-3C=-1.$$

Solving for A, B, C, and D, we get $A=B=2$ and $C=-1$. Finally, we compute for the integral.

$$\begin{aligned}
\int \frac{4x^2-x-1}{(x+1)^2(x-3)}\,dx &= \int \frac{2}{x-3}\,dx+\int \frac{2}{x+1}\,dx-\int \frac{1}{(x+1)^2}\,dx \\
&= 2\ln|x-3|+2\ln|x+1|+\frac{1}{x+1}+C
\end{aligned}$$

□